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Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.4,pp.413-422,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

A GAME OF OPTIMAL PURSUIT OF ONE NON-INERTIAL OBJECT BY TWO INERTIAL OBJECTS*

A.YU. LEVCHENKOV and A.G. PASHKOV

A game in which one controlled object is pursued by two others is studied. The pursuing objects are inertial, and the pursued object is not. The duration of the game is fixed. The payoff functional is the distance between the pursued object and the closest pursuer at the instant when the game ends. An algorithm for determining the payoff function for all possible positions is constructed. It is shown that the game space consists of several domains in which the payoff is expressed analytically, or is determined by solving a certain non-linear equation. Strategies of the pursuers which guarantees them a result as close to the game payoff as desired are indicated.

The optimal solution of a game of pursuit when one inertial object pursues a non-inertial one was obtained earlier in /1/. The present paper is related to the investigations reported in /1—10/.

1. Let the motions of the pursuers $P_i(x^i)$ $(i = 1, 2)$ and of the pursued object $E(z)$ be described by the equations

$$x_1^{\cdot i} = x_3^i, \quad x_3^{\cdot i} = u_1^i, \quad x_2^{\cdot i} = x_4^i, \quad x_4^{\cdot i} = u_2^i, \quad z_1^{\cdot} = v_1, \quad z_2^{\cdot} = v_2 \tag{1.1}$$

The control vectors of the pursuers and the pursued satisfy the constraints

$$((u_1^i)^2 + (u_2^i)^2)^{1/2} \leqslant \mu > 0, \quad (v_1^2 + v_2^2) \leqslant \nu \tag{1.2}$$

The game is studied over the time interval $[t_0, \vartheta)$. The payoff functional is the distance between the pursued object and the nearest pursuer at the instant $t = \vartheta$ that the game ends, i.e.

$$\gamma = \min_i [(z_1(\vartheta) - x_1^i(\vartheta))^2 + (z_2(\vartheta) - x_2^i(\vartheta))^2]^{1/2} \tag{1.3}$$

As a result of the change of variables $y_j^i = x_j^i + (\vartheta - t)\, x_{j-2}^i$ $(j = 1, 2)$, which means passing to considering the centres of regions of attainability of the inertial objects, relations (1.1)–(1.3) take the form

$$y_j^{\cdot i} = (\vartheta - t)\, u_j^i, \quad y_j^i(t_0) = x_j^i(t_0) + (\vartheta - t_0)\, x_{j-2}^i(t_0) \tag{1.4}$$

$$\gamma = \min_i [(z_1(\vartheta) - y_1^i(\vartheta))^2 + (z_2(\vartheta) - y_2^i(\vartheta))^2]^{1/2} \tag{1.5}$$

At the instant $t = \vartheta$ the values of γ found from (1.3) and (1.5) are identically equal.

We denote the centres of the attainability regions by P_i. For the positions where $P_1^c = P_2^c$, the payoff of the two-to-one game, denoted by ρ^{21}, is identical with the payoff of the one-to-one game denoted by ρ^{11}. Henceforth we consider those initial positions for which $P_1^\circ \neq P_2^\circ$.

*Prikl.Matem.Mekhan.,49,4,536-547,1985

Let us introduce a mobile coordinate system linked to the current position of the pursuers. We direct the q_1 axis from the current position of the first pursuer to the current position of the second (the numbering of the pursuers is fixed and arbitrary). The q_2 axis runs through the middle of the segment $[P_1P_2]$, at right angles, so as to obtain a right-oriented system of coordinates. In this system, the position of the object E will be defined by the coordinates $\{x, y\}$, and that of the pursuers P_i by the coordinates $\{(-1)^{i+1}z, 0\}$. Because the position of the pursuers is symmetric in this system, the vector $\xi(x, y, z)$ fully describes the mutual location of the pursuers and the pursued.

In special cases, simultaneously with the above mobile coordinate system (q_1, q_2) we shall consider an immobile Cartesian system (η_1, η_2), the axes of both systems coinciding at a certain instant of time. The system (η_1, η_2) is convenient for carrying out geometrical constructions and for considering optimal motions.

The dynamics of the phase vector ξ is described by the following system of differential equations:

$$x^{\cdot} = v_1 - \frac{(\vartheta - t)}{2}[u_1^1 + u_1^2] + \frac{y(\vartheta - t)}{2z}[u_2^2 - u_2^1] \tag{1.6}$$

$$y^{\cdot} = v_2 - \frac{(\vartheta - t)}{2}[u_2^1 + u_2^2] - \frac{x(\vartheta - t)}{2z}[u_2^2 - u_2^1]$$

$$z^{\cdot} = \frac{(\vartheta - t)}{2}[u_1^2 - u_1^1]$$

The constraints on the control of the players have the form (1.2). The payoff functional is determined from the formula

$$\gamma = [(z(\vartheta) - |x(\vartheta)|)^2 + y^2(\vartheta)]^{1/2} \tag{1.7}$$

In (1.6), the vector $v = \{v_1, v_2\}$ has, in relation to the system (η_1, η_2)), the meaning of the absolute velocity of the point E, and the vectors $u^i = \{u_1^i, u_2^i\}$ are proportional, with a factor $(\vartheta - t)$, to the velocities of the points P_i. Thus the first two formulae in (1.6) produce expressions for the relative velocity of the point E in the mobile system (q_1, q_2), and the component $z^{\cdot}$ describes the relative velocity of the pursuer.

We shall carry out some geometrical constructions in the coordinate system (η_1, η_2). A circle of radius $r(t_0) = \mu(\vartheta - t_0)^2/2$ with its centre at the point $\{(-1)^{i+1}z(t_0), 0\}$ will be the attainability region $G^i\{t_0, \vartheta\}$ of object P_i from the specified initial position at the instant $t = t_0$ to the instant $t = \vartheta$. The attainability domain $G_e(t_0, \vartheta)$ of object E from the specified initial location position at the instant $t = t_0$ to the instant $t = \vartheta$ will be a circle of radius $R(t_0) = \nu(\vartheta - t_0)$ with its centre at the point $\{x, y\}$. We shall denote the boundaries of the domains $G^i(t_0, \vartheta)$ and $G_e(t_0, \vartheta)$ by $\partial(G^i)$ and $\partial(G_e)$ respectively. We shall mean by the position of a game the vector $\{t, \xi(t)\}$ of the extended phase space.

Suppose that we are given $\{t_0, \xi(t_0)\}$ as the initial position of the game. The following mutual locations of the objects P_i and E, the attainability region $G_e(t_0, \vartheta)$, and the η_2 axis are possible:

1) $\partial(G_e) \cap \eta_2 = \{\varnothing\}$ or $\partial(G_e) \cap \eta_2 = \{A\}$, where A is a unique point;
2) $\partial(G_e) \cap \eta_2 = \{A_1, A_2\}$ with $A_1 \neq A_2$ and $E \not\subseteq P_1A_1P_2A_2$;
3) $\partial(G_e) \cap \eta_2 = \{A_1, A_2\}$ with $A_1 \neq A_2$ and $E \in \text{int}\, P_1A_1P_2A_2$.

The first two cases are described by the following inequality containing the vector ξ and time:

$$\frac{|x|}{\nu(\vartheta - t)} \geqslant \frac{z - |x|}{((z - |x|)^2 + y^2)^{1/2}} \tag{1.8}$$

The situation corresponding to case 3) is described by the opposite inequality and is shown in Fig.1.

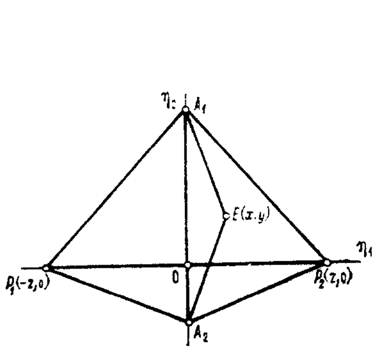

Fig.1

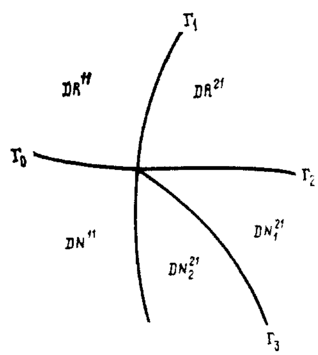

Fig.2

2. Inequality (1.8) separates out in phase space a certain three-dimensional domain (denoted by D^{11}), in which the one-to-one game, that is $\rho^{21} = \rho^{11}$ takes place. Obviously in D^{11} the problem reduces to a game of pursuit between E and the nearest pursuer.

Let us divide the domain D^{11} into subdomains DR^{11} and DN^{11}. We consider the quadratic equation

$$(t-t_0)^2 - 2(t-t_0)\left(\vartheta - t_0 - \frac{\nu}{\mu}\right) + \frac{2c}{\mu} = 0 \tag{2.1}$$
$$c = ((|x| - z)^2 + y)^2)^{1/2}$$

The boundary Γ_0 of DR^{11} and DN^{11} satisfies the relations

$$d = \left(\vartheta - t_0 - \frac{\nu}{\mu}\right)^2 - \frac{2c}{\mu} = 0, \quad t_1 = t_2 \geqslant t_0 \tag{2.2}$$

(d is the discriminant, and t_1, t_2 are the roots of Eq.(2.1)).

One of the following two conditions is satisfied in domain DR^{11}: either $d < 0$, or $d > 0$ and $t_1 < t_2 < t_0$. In domain DN^{21} the real roots of Eq.(2.1) satisfy the inequality $t_0 < t_1 < t_2$.

We denote by γ_*^{11} the programmed maximin in the one-to-one game. It follows from /1/ that in the domain DR^{11} the payoff of a game satisfies the relation $\rho^{11} = \gamma_*^{11}$, and in the domain $\overline{DN^{11}} = DN^{11} \cup \Gamma_0$ the equation $\rho^{11} = \nu^2/(2\mu)$. Obviously, for $t_0 > \vartheta - \nu/\mu$ the set $\overline{DN^{11}}$ is empty.

3. Consider case 3) shown in Fig.1 (in (1.8) there is an opposite inequality). This case is comparable with the three-dimensional domain D^{21} separated from D^{11} by the surface Γ_1 defined by the relation $E \in \partial(P_1A_1P_2A_2)$, where $\partial(P_1A_1P_2A_2)$ is the boundary of the tetragon $P_1A_1P_2A_2$. The surface Γ_1 consists of three parts: ΓR_1 (this separates DR^{11} and D^{21}), ΓN_1 (this separates DN^{11} and D^{21}), and the line L on which the relations (2.2) are satisfied together with the condition $E \in \partial(P_1A_1P_2A_2)$.

Let us divide the domain D^{21} into the open subdomains DR^{21} and DN^{21}. For this, we consider the quadratic equation

$$(t-t_0)^2 - 2(t-t_0)\left(\vartheta - t_0 - \frac{\nu \sin\alpha_0}{\mu \sin\beta_0}\right) + \frac{2y}{\mu\sin\beta_0} = 0 \tag{3.1}$$
$$\sin\alpha_0 = (\nu^2(\vartheta - t_0)^2 - x^2(t_0))^{1/2}/(\nu(\vartheta - t_0))$$
$$\sin\beta_0 = \frac{y(t_0) + x(t_0)\,\mathrm{tg}\,\alpha_0}{((y(t_0) + x(t_0)\,\mathrm{tg}\,\alpha_0)^2 + z^2(t_0))^{1/2}}$$

The surface Γ_2 will be a boundary of subdomains DR^{21} and DN^{21}. The points of Γ_2 satisfy the relations

$$d^* = \left(\vartheta - t_0 - \frac{\nu\sin\alpha_0}{\mu\sin\beta_0}\right)^2 - \frac{2y}{\mu\sin\beta_0} = 0, \quad t_1 = t_2 \geqslant t_0 \tag{3.2}$$

(d^* is the discriminant, and t_1, t_2 are the roots of Eq.(3.1)).

Let us clarify the meaning of Eq.(3.1). Let points A_1 and A_2 in the fixed system (η_1, η_2) have the coordinates $(0, a_1)$ and $(0, a_2)$ respectively, and point A the coordinates $(0, \max_i\{|a_i|\}$ sign $y(t_0))$ (i.e., A is the point of the set $\{A_1, A_2\}$ furthest removed from the pursuers). Next, let the players P_i and E take extremal aim at the point A. We shall describe the corresponding motion as an extremal programmed motion. Then the number $t = t_1$ will be the root of Eq.(3.1) if on the extremal programmed motion we have $y(t_1) = 0$. Thus, the presence of the root $t = t_1$ reflects the fact that the projections of points P_i and E coincide on the η_2 axis at the instant $t = t_1$.

The domain DR^{21} is a subdomain D^{21} in which one of the following conditions is satisfied: $d^* < 0$ or $d^* > 0$, but neither of the roots t_1 and t_2 exceeds t_0. The domain DN^{21} is a subdomain D^{21} in which the condition $d^* > 0$, and $t_1 > t_0$, $t_2 > t_0$ hold. The cases described cover all possible relations of the roots t_1, t_2 of Eq.(3.1), and t_0 since the situation $t_1 < t_0 < t_2$ is impossible because of the definition of point A. Therefore we have $DR^{21} \cup \Gamma_2 \cup DN^{21} = D^{21}$.

The points of the surface Γ_2 have the following property. By virtue of Eq.(3.2), the roots of Eq.(3.1) are identical; at the same time, at the instant $t = t_1 = t_2$ not only the projections but also the velocities of projections of points P_i and E on the η_2 axis are equal.

The division of the phase space is shown schematically in Fig.2.

4. Let $\{t_0, \xi(t_0)\} \in DR^{21} \cup \Gamma_2 = \overline{DR^{21}}$. We shall consider the function of programmed maximin.

$$\gamma_*^{21} = \max\{\gamma_1, \gamma_2\} \tag{4.1}$$
$$\gamma_k = (z^2(t_0) + a_k^2)^{1/2} - \mu(\vartheta - t_0)^2/2, \; k = 1, 2$$
$$a_{1,2} = y(t_0) \pm ((\nu(\vartheta - t_0))^2 - x^2(t_0))^{1/2}$$

It can be shown that γ_*^{21} is u-stable /2/ in the domain $\overline{DR^{21}}$. The property of v-stability of the function γ_*^{21} follows from the definition of this function. Thus, a function with (u, v) stability has been constructed in the domain $\overline{DR^{21}}$.

Proof of the u-stability of γ_*^{21} in the domain $\overline{DR^{21}}$. We introduce additional constraints on the control of the players P_i. We set

$$u_1^1 = -u_1^2 = u_1, \quad u_2^1 = u_2^2 = u_2 \tag{4.2}$$

System (1.6) takes the form

$$x^{\cdot} = v_1, \quad y^{\cdot} = v_2 - (\vartheta - t)\, u_2, \quad z^{\cdot} = -(\vartheta - t)\, u_1 \tag{4.3}$$

Let us show that when the constraints (4.2) are imposed on the control of the pursuers, the function γ_*^{21} will be u-stable in $\overline{DR^{21}}$. Hence follows the u-stability of this function also when there are no constraints.

1°. Let $\{t_0, \xi(t_0)\} \in \overline{DR^{21}}$ with $y(t_0) > 0$. This means that $E \notin [P_1 P_2]$. In this case under consideration we have $\gamma_*^{21} = \gamma_1 > \gamma_2$. We shall prove that under these conditions the function γ_*^{21} satisfies the Bellman equation

$$\frac{\partial \gamma_*^{21}}{\partial t} + \max_v \min_u \left[\left(\frac{\partial \gamma_*^{21}}{\partial x}\right) x^{\cdot} + \left(\frac{\partial \gamma_*^{21}}{\partial y}\right) y^{\cdot} + \left(\frac{\partial \gamma_*^{21}}{\partial z}\right) z^{\cdot}\right] = 0 \tag{4.4}$$

We introduce the notation

$$r = ((\nu(\vartheta - t))^2 - x^2)^{1/2}, \quad a_1 = y + r, \quad R_1 = (z_1^2 + a_1^2)^{1/2}$$

Then $\gamma_*^{21} = R_1 - \mu(\vartheta - t)^2/2$. On substituting this expression into (4.4), we obtain

$$\max_v \min_u \left\{\frac{d\gamma_*^{21}}{dt}\right\} = \min_u \left(-\frac{(\vartheta - t)\, u_1 z}{R_1} - \frac{(\vartheta - t)\, u_2 a_1}{R_1}\right) +$$
$$\max_v \left(-\frac{a_1 x v_1}{R_1 r} + \frac{a_1 v_2}{R_1}\right) + \mu(\vartheta - t) - \frac{a_1 \nu^2 (\vartheta - t)}{R_1 r}$$

It can be verified that the minimum with respect to u on the right side of this expression equals $-\mu(\vartheta - t)$, and the maximum with respect to v is $-a_1 \nu^2 (\vartheta - t)/(R_1 r)$. Thus the basic equation is satisfied.

2°. Now let $\{t_0, \xi(t_0)\} \in \overline{DR^{21}}$ with $y(t_0) = 0$. Hence, for $t = t_0$ the equality $\gamma_*^{21} = \gamma_1 = \gamma_2$ holds, and the function γ_*^{21} is not differentiable. We use Theorem (3.2.1) from /3/ to vertify the u-stability of the function γ_*^{21}. Thus, we must prove the inequality

$$\max_v \min_u \max \{d\gamma_1/dt, d\gamma_2/dt\} \leqslant 0 \tag{4.5}$$

Let us introduce the following notation:

$$r = ((\nu(\vartheta - t))^2 - x^2)^{1/2}, \quad R = (z^2 + r^2)^{1/2}$$

Then $\gamma_*^{21} = R - \mu(\vartheta - t)^2/2$, $a_1 = r$, $a_2 = -r$, and inequality (4.5) takes the form

$$R^{-1} \max_v \min_u \{-(\vartheta - t)\, u_1 z - x v_1 + r\,|v_2 - (\vartheta - t)\, u_2|\} + \mu(\vartheta - t) - \frac{\nu^2(\vartheta - t)}{R} \leqslant 0 \tag{4.6}$$

We note that the validity of the inequalities $\nu \geqslant \mu(\vartheta - t)\, r R^{-1}$, $R \geqslant \mu(\vartheta - t)^2$ follows from the condition $\{t_0, \xi(t_0)\} \in \overline{DR^{21}}$.

Consider the function

$$\varphi(x, z, u, v) = -(\vartheta - t)\, u_1 z - x v_1 + r\,|v_2 - (\vartheta - t)\, u_2|$$

To estimate the function φ, we consider its contour lines $\varphi = c = \text{const}$ in (s_1, s_2) axes where $s_1 = u_1(\vartheta - t)$, $s_2 = u_2(\vartheta - t)$. We denote by δ the straight line $u_2(\vartheta - t) = v_2$ in the (s_1, s_2) plane. Let $v_2 \in [\mu(\vartheta - t)\, r R^{-1}, \nu]$. Under this assumption a minimum is attained at the point A (Fig.3a).

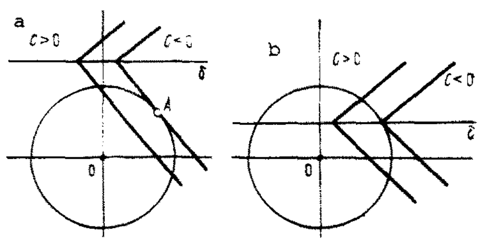

Fig.3

Then we have

$$\min_u \varphi(x, z, u, v) = r v_2 - x v_1 - R\mu(\vartheta - t).$$

Finally, we obtain

$$\max_v \min_u \varphi = \max_v (r v_2 - x v_1 - R\mu(\vartheta - t)) = \nu^2(\vartheta - t) - R\mu(\vartheta - t)$$

Clearly, expression (4.6) holds (equality occurs).

Now, let $0 \leqslant v_2 \leqslant \mu(\vartheta - t)\, r R^{-1}$. Then the minimum of the function φ with respect to u is attained for $u_2 = v_2/(\vartheta - t)$ (Fig.3b). Therefore,

$$\max_v \min_u (-(\vartheta - t) u_1 z - x v_1 + r \mid v_2 - (\vartheta - t) u_2 \mid) = \max_v (-z (\mu^2 (\vartheta - t)^2 - v_2^2)^{1/2} - x v_1)$$

To be specific, let us assume that $x \geqslant 0$. Then, obviously, $v_1 = -(v^2 - v_2^2)^{1/2}$. Consider the function $f(v_2) = x(v^2 - v_2^2)^{1/2} - z(\mu^2(\vartheta - t)^2 - v_2^2)^{1/2}$.

By computing the derivative f_{v_2}' we can show that, under the condition $R \geqslant \mu(\vartheta - t)^2$, the function $f(v_2)$ increases monotonically in the section $0 \leqslant v_2 \leqslant \mu(\vartheta - t) rR^{-1}$. Therefore, the maximum of $f(v_2)$ is attained at the end point of the section when $v_2 = \mu(\vartheta - t) rR^{-1}$, and inequality (4.6) becomes a strict equality.

Thus, inequality (4.5) is proved.

The case of $v_2 < 0$ can be examined similarly.

Note that the proof of the u-stability of the function γ_*^{21} for the case in Sect.2° could have been constructed in the same way as the proof given in /10/.

5. Consider the set DN^{21}. For $t \geqslant \vartheta - \nu/\mu$ we have $DN^{21} = \{\varnothing\}$. Therefore, for the points of set $\overline{DN^{21}}$ the inequality $t < \vartheta - \nu/\mu$. It can be shown that

$$\min_{\{t, \xi\}} \rho^{21}(t, \xi) = \nu^2/(2\mu), \quad \{t, \xi\} \in DN^{21} \tag{5.1}$$

Clearly, $\rho^{21} = \nu^2/(2\mu)$ corresponds, for example, to those positions of $\{t, \xi(t)\} \in DN^{21}$ where absorption occurs (S_2 is the two-dimensional sphere of unit radius, $k = \nu^2/(2\mu)$)

$$\{G^i(t, \vartheta)\} \oplus kS_2 \supset \{G_e(t, \vartheta)\}$$

The relations

$$\min_{\{t, \xi\}} \rho^{21} = \nu^2/(2\mu), \ \{t, \xi\} \in \Gamma N_1 \cup L; \qquad \inf_{\{t, \xi\}} \gamma_*^{21} = \nu^2/(2\mu), \ \{t, \xi\} \in \Gamma_2$$

hold for any $t \leqslant \vartheta - \nu/\mu$.

Since $DN^{21} = \{\varnothing\}$ when $t = \vartheta - \nu/\mu$. the trajectory of the system should, starting from any initial position $\{t_0, \xi(t_0)\} \in DN^{21}$ when $t_0 < \vartheta - \nu/\mu$, cross either Γ_1 or Γ_2 not later than the instant $t = \vartheta - \nu/\mu$. Therefore, Eq.(5.1) holds.

Let us divide the set DN^{21} into two: DN_1^{21}, where $\rho^{21} > \nu^2/(2\mu)$, and DN_2^{21}, where $\rho^{21} = \nu^2/(2\mu)$. We will denote the boundary of domains DN_1^{21} and DN_2^{21} by Γ_3. An algorithm for constructing these domains is given in Sect.11.

6. Let $\{t_0, \xi(t_0)\} \in DN_1^{21}$. For this position we formulate auxiliary game problem 1 whose conditions are as follows:

A. The equations of motion and the constraints on the control of the players are identical with (1.6) and (1.2).

B. The time of the game, T, is not fixed (it follows from Sect.5 that $T \leqslant \vartheta - \nu/\mu$).

C. The payoff of Game 1 will be the value of ρ^{11}, if the system trajectory has emerged at boundary Γ_1, or the value of γ_*^{21} if it emerged at boundary Γ_2.

To solve Game 1 it is necessary to consider the auxiliary Games 2 and 3 formulated below.

7. Let $\{t_0, \xi(t_0)\} \in DN_1^{21}$, with $E \in [P_1 P_2]$, i.e. $y(t_0) = 0$. We introduce the auxiliary Game 2 by the following conditions:

A. The equations of motion of the players are identical with (1.6).

B. Besides the constraints (1.2), the following constraint is imposed on the control of the pursuers: for $t_0 \leqslant t \leqslant T$, the relation $y(t) = 0$ should hold along the system trajectory.

C. The instant T of the end of the game is not fixed.

D. The payoff of the game and the conditions of its termination are as Condition C in Sect.6.

We note that in setting Game 2, the class of admissible strategies of the pursuers was changed: condition B can be satisfied in the class of counterstrategies of players P_i only. In this case the result of the initial game will not change because, for a problem involving the dynamics which is described by Eq.(1.6), a saddle point exists in the 'little' game.

We will show that Game 2 will end on the surface Γ_2.

We assume the contrary, i.e. that the phase trajectory has crossed the boundary Γ_1: $\{t, \xi(t)\} \in \Gamma_1$. This, together with the condition $y(t) = 0$, means that at the instant t, player E coincided with one of the pursuers: $E = P_i$. But the payoff of the game for such a position is $\rho^{21} = \rho^{11} = \nu^2/(2\mu)$. Thus we arrive at a constradiction since the payoff of the game at the initial position is $\rho^{21} > \nu^2/(2\mu)$.

Consider the following strategies of the palyers:

$$\begin{aligned} &U_0^{(2)}: \ u_1^1 = -u_1^2 = (\mu^2 - (u_2^1)^2)^{1/2}, \quad u_2^1 = u_2^2 = v_2/(\vartheta - t) \\ &V_0^{(2)}: \ v_1 = -\operatorname{sign}(x) \min\left\{|x| \left(\frac{\mu^2(\vartheta - t)^2 - \nu^2}{z^2 - x^2}\right)^{1/2}, \ \nu\right\} \\ &v_2 = (\nu^2 - v_1^2)^{1/2} \end{aligned} \tag{7.1}$$

($U_0^{(2)}$ is the countercontrol of the pursuers, and $V_0^{(2)}$ denotes the positional control of the evader).

Also, let $U^{(2)}(\xi, v)$ be an arbitrary counterstrategy of the pursuers, $V^{(2)}(\xi)$ an arbitrary positional control of the evader, and γ^* the value of the functional in Game 2 on the corresponding strategies.

It can be shown that for $U_0^{(2)}$ and $V_0^{(2)}$, the inequality of the saddle point

$$\gamma^*(U_0^{(2)}, V^{(2)}) \leqslant \gamma^*(U_0^{(2)}, V_0^{(2)}) \leqslant \gamma^*(U^{(2)}, V_0^{(2)}) \tag{7.2}$$

holds.

1°. First we shall prove the left-hand inequality of (7.2). To do this, we substitute strategy $U_0^{(2)}$ into Game 2, and obtain a problem of the optimal control of player E, of the form

$$x^\cdot = v_1, \quad z^\cdot = -(\mu^2(\vartheta - t)^2 - v_2^2)^{1/2}, \quad (v_1^2 + v_2^2)^{1/2} \leqslant \nu \tag{7.3}$$

For $E \in [P_1P_2]$ on the section $t_0 \leqslant t \leqslant T$, the condition of ending the game (reaching Γ_2 by the trajectory) takes the form

$$\Phi = (\nu^2(\vartheta - T)^2 + z^2(T) - x^2(T))^{1/2} - \mu(\vartheta - T)^2 = 0 \tag{7.4}$$

The programmed maximin is found from the formula

$$\gamma_*^{21} = (\nu^2(\vartheta - T)^2 + z^2(T) - x^2(T))^{1/2} - \mu(\vartheta - T)^2/2 \tag{7.5}$$

Thus, the functional of the problem is $\gamma^* = \max_{U^{(2)}} \gamma_*^{21}(T)$.

By the definition of the domain DN_1^{21}, if the initial position is $\{t_0, \xi(t_0)\} \in DN_1^{21}$ and the inequality $z(t_0) > |x(t_0)| > 0$ holds, the analogous inequality will hold at the instant $t = T$: $z(T) > |x(T)| \geqslant 0$. It can be shown that for $t_0 \leqslant t \leqslant T$ the identity $x(t) \equiv 0$ follows from the equation $x(t_0) = 0$.

It follows from the maximum principle /11/ that the optimal control of player E in Game 2 should have the form

$$v_1 = -\operatorname{sign}(x(T)) |\min\left\{|x|\left(\frac{\mu^2(\vartheta - t) - \nu^2}{z^2(T) - x^2(T)}\right), \nu\right\} \tag{7.6}$$
$$v_2 = \pm(\nu^2 - v_1^2)^{1/2}$$

Clearly, this value of the functional does not depend on the sign of the control v_2 since the trajectories generated by these controls are symmetric with respect to the η_1 axis. To be specific, let us set $v_2 \geqslant 0$, and analyse the expression for v_1 from (7.6). We shall assume that for small t, a minimum is attained at the second term, that is $v_1(t) = -\operatorname{sign}(x(T))\nu$. In the coordinate system (η_1, η_2), the rectilinear sections of the players' trajectories correspond to this control (Fig.4). Starting at a certain instant $t = t_*$ up to the instant $t = T$, a minimum in (7.6) will be attained at the first term. Therefore player E makes use of the control

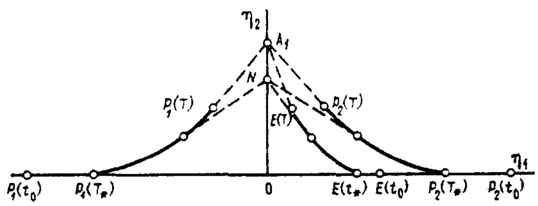

Fig.4

$$v_1(t) = -x(T)(\mu^2(\vartheta - t)^2 - \nu^2)^{1/2} / (z^2(T) - x^2(T))^{1/2} \tag{7.7}$$

On substituting Eq.(7.7) into system (7.3) we find that for $t \in [t_*, T]$ the relations

$$z^\cdot(t) / x^\cdot(t) = z(T) / x(T) = z(t) / x(t) \tag{7.8}$$

exist, i.e. $x(T), z(T)$ can be replaced by $x(t), z(t)$ in the control law.

The equality

$$x(t_*)(\mu^2(\vartheta - t_*)^2 - \nu^2)^{1/2} / (z^2(t_*) - x^2(t_*))^{1/2} = \nu$$

holds at the instant $t = t_*$, hence $z(t_*) / (\mu(\vartheta - t_*)) = x(t_*)/\nu$. In a fixed coordinate system the beginning of a curvilinear motion by the players will correspond to the instant $t = t_*$ (Fig.4).

It follows from (7.7) and (7.8) that on the curvilinear trajectory segment the projections of the velocities of the players P_i and E on the η_1 axis are proportional to the phase coordinates. For the velocity projections on the η_2 axis we have $(\vartheta - t)u_2^1 = (\vartheta - t)u_2^2 = v_2$. Hence, considering (7.7) and (7.8), we have

$$-x(t)v_2(t) / v_1(t) = z(t)u_2^1(t) / u_1^1(t)$$

This means that on the curvilinear parts of the trajectory the velocity vectors of players P_i and E are directed at the same point N lying on the η_2 axis of the fixed coordinate system.

During the curvilinear motion the point N shifts along the η_k axis from the point O for $t = t_*$ to the point A (the point of extremal aiming) for $t = T$.

The problem of optimal control (7.3)–(7.5) is solved.

2°. The control (7.6) which solves problem (7.3)–(7.5) is a programmed control. However, using (7.7) and (7.8) this problem can be rewritten in a form identical to $V_0^{(2)}$, and considered as the positional control of player E.

Let us now prove the first inequality in (7.2). For this, we substitute $V_0^{(2)}$ into Game 2, thus obtaining a problem of optimal control for players P_i, of the form

$$x' = v_1(t, x, z) - \eta_1^+, \; z' = \eta_1^-; \; ((u_1^i)^2 + (u_2^i)^2)^{1/2} \leqslant \mu \tag{7.9}$$

The equation relating the phase coordinates and controls is given by

$$\Psi = v_2(t, x, z) - \eta_2^+ - (x/z)\eta_2^- = 0 \tag{7.10}$$

The condition for ending the process and the payoff are (7.4) and (7.5) respectively (the pursuers tend to minimize $\gamma_*^{21}(T)$).

In Eqs.(7.9) and (7.10) the functions $v_k(t, x, z)$ are components of the positional strategy $V_0^{(2)}$ of player $E, \eta_k^\pm = {}^1/_2(\vartheta - t)[u_k^2 \pm u_k^1]$. $k = 1, 2$.

As was done in 1°, a check is made that the programmed control of the pursuers $u^{(2)}(t) = U_0^{(2)}(V_0^{(2)})$ satisfies the maximum principle for problem (7.9), (7.10), (7.4), (7.5).

8. Let $\{t_0, \xi(t_0)\} \in DN^{21}$ and $E \in [P_1P_2]$, that is $y(t_0) = 0$. We shall present an algorithm for obtaining the functional γ^* of the auxiliary Game 2. We shall assume, to be specific, that $x(t_0) \geqslant 0$. Also, let the inequality $z(t_0)/(\mu(\vartheta - t_0)) \geqslant x(t_0)/\nu$ which implies that in the optimal trajectory of Game 2 there is no straight line section, be satisfied at the instant $t = t_0$ (Fig.4). We introduce the notation

$$J(t_0, t) = \int_{t_0}^{t} (\mu^2(\vartheta - \tau)^2 - \nu^2)^{1/2} d\tau, \quad \Delta_0 = (z^2(t_0) - x^2(t_0))^{1/2}$$

The equalities

$$x(t) = x(t_0)(1 - J(t_0, t))/\Delta_0, \; z(t) = z(t_0)(1 - J(t_0, t))/\Delta_0 \tag{8.1}$$

hold on the curvilinear section.

Consider the equation

$$J(t_0, t)/\Delta_0 = 1 \tag{8.2}$$

If it has the root $t = t_* \in [t_0, \vartheta]$, then at the instant $t = t_*$ the equality $z(t_*) = x(t_*) = 0$ holds. This points to the fact that $\gamma^* = \nu^2/(2\mu)$, and that the initial position is $\{t_0, \xi(t_0)\} \in DN_2^{21}$, i.e. $\rho^{21} = \nu^2/(2\mu)$.

Suppose that Eq.(8.2) has no root. At the instant $t = T$ Eq.(7.4) should be satisfied. On substituting (8.1) into (7.4), we obtain the following non-linear equation for determining the time T of Game 2:

$$[\nu^2(\vartheta - T)^2 + (1 - J(t_0, T)/\Delta_0)^2 \Delta_0^2]^{1/2} = \mu(\vartheta - T)^2 \tag{8.3}$$

We find the functional from the formula $\gamma^* = \mu(\vartheta - T)^2/2$.

If the relation $z(t_0)/(\mu(\vartheta - t_0)) < x(t_0)/\nu$ holds at the instant $t = t_0$, it means that the optimal trajectory of Game 2 has a straight-line section. Therefore we first seek the minimum root $t = t^*$ of the quadratic equation $z(t)/(\mu(\vartheta - t)) = x(t)/\nu$, where

$$z(t) = z(t_0) - \int_{t_0}^{t} \mu(\vartheta - \tau) d\tau, \quad x(t) = x(t_0) - \nu(t - t_0)$$

It can be shown that for $\{t_0, \xi(t_0)\} \in DN_1^{21}$ this root (which corresponds to the instant when the straight-line section ends) certainly exists. Further, we assume $t_0 = t^*$ and perform the operations given at the beginning of this section.

9. Now let $\{t_0, \xi(t_0)\} \in DN_1^{21}$ with $E \notin [P_1P_2]$. Let us assume that $y(t_0) > 0$, and formulate auxiliary Game 3 for the above position.

A. The equations of motion and the constraints on the control of players are identical with (1.6).

B. The time of game T_f is not fixed.

C. Pursuers P_i tend to lead out the trajectory of the system on the surface $\Psi \equiv y(T_f) = 0$, at the same time minimizing the payoff $\gamma^*(T_f)$. The problem of the evader is the opposite.

Below we shall build the positional strategies of the players which yield a saddle point for Problem 3.

By the relations $\gamma^* = \gamma^*(x, z, t), \Psi = \Psi(y)$ and the maximum principle /11/, the equations

$$\frac{d}{dt}\Psi\Big|_{t=T_f} = 0, \quad \frac{d}{dt}\gamma^*\Big|_{t=T_f} = 0 \tag{9.1}$$

should hold on the terminal surface $\Psi(y) = 0$. These equations express the fact that the trajectory approaches the terminal surface along a tangential line. Suppose that the point $N(0, n)$ belongs to the η_2 axis. We shall consider the coordinate n as a parameter and use the notation

$$U_N^{(3)} = \{(-1)^{i-1} z\mu (z^2 + n^2)^{-1/2},\ n\mu (z^2 + n^2)^{-1/2}\}$$

$$V_N^{(3)} = \{-x\nu (x^2 + (n-y)^2)^{-1/2},\ (n-y)\nu (x^2 + (n-y)^2)^{-1/2}\}$$

($U_N^{(3)}$ and $V_N^{(3)}$ are the extremal controls of players P_i and E, oriented at the point N).

We define the value of the parameter $n = n^*$ such that in the motion generated by the controls $U_{N^*}^{(3)}$ and $V_{N^*}^{(3)}$ at the instant the game terminates $t = T_f$, the conditions of tangency (9.1) are satisfied, that is $\Psi^{\cdot} \equiv y^{\cdot} = 0$. Since the initial position $\{t_0, \xi(t_0)\}$ belongs to the set DN_1^{21}, the desired value of the parameter n^* and the corresponding point $N^*(0, n^*)$ certainly exist and are unique (because $E \notin [P_1 P_2]$ when $t = t_0$). Using the maximum principle one can check that $U_{N^*}^{(3)}$ and $V_{N^*}^{(3)}$ yield a saddle point of the auxiliary Game 3:

$$\gamma^{**}(U_{N^*}^{(3)},\ V^{(3)}) \leqslant \gamma^{**}(U_{N^*}^{(3)},\ V_{N^*}^{(3)}) \leqslant \gamma^{**}(U^{(3)},\ V_{N^*}^{(3)}) \tag{9.2}$$

(γ^{**} is the value of the payoff γ^* on the corresponding strategies).

10. Given the initial position $\{t_0, \xi(t_0)\} \in DN_1^{21}$, let us describe an algorithm for obtaining γ^{**}. We put

$$\sin\alpha_1 = (n - y(t_0)) / ((n - y(t_0))^2 + x^2(t_0))^{1/2}$$
$$\sin\beta_1 = n / (n^2 + z^2(t_0))^{1/2} \tag{10.1}$$

and consider the equation

$$\sin\beta_1 \int_{t_0}^{T_f} \mu(\vartheta - \tau)\, d\tau = \nu(T_f - t_0)\sin\alpha_1 + y(t_0) \tag{10.2}$$

On substituting (10.1) into (10.2) we obtain a quadratic equation with parameter n relative to the time T_f when auxiliary Game 3 ends. Then $n = n^*$ is the desired value of the parameter if the discriminant of Eq.(10.2) is zero for $n = n^*$. The instant $t = T_f$ when Game 3 ends corresponds to the value of n^*, and the game's final position $\{T_f, x(T_f), 0, z(T_f)\}$ is an initial position for the auxiliary Game 2. Applying the procedure described in Sect.8, we obtain the value of the programmed maximin γ_*^{21} at the instant when the trajectory appears on the boundary Γ_2, $t = T$. We assume that $\gamma^{**} = \gamma_*^{21}(T)$.

11. For $t = t_0$, using the positional strategies $U_{N^*}^{(3)}$ and $V_{N^*}^{(3)}$ we can divide the set DN^{21} into domains DN_1^{21} and DN_2^{21}. Set DN_1^{21} consists of a position $\{t_0, \xi(t_0)\}$ for which the algorithm from Sect.10 yields $\gamma^{**} > \nu^2\ (2\mu)$. On set DN_2^{21} the equation

$$\gamma^{**} = \nu^2/(2\mu) \tag{11.1}$$

holds.

For the points of set DN_1^{21}, the strategies which furnish (11.1) are unique for both players. This follows from inequality (9.2). At the boundary Γ_3 of domains DN_1^{21} and DN_2^{21} the strategies of the pursuers, which ensure for them the existence of (11.1), are unique, but the evader's strategy is not unique. At the inner points of domain DN_2^{21} the strategies of both sides are not unique. This phenomenon takes place in domain D^{11} as well (see /1/).

12. We set

$$\gamma^{***} = \begin{cases} \gamma_*^{21}, & \{t_0, \xi(t_0)\} \in \overline{DR^{21}} \\ \gamma^{**}, & \{t_0, \xi(t_0)\} \in DN^{21} \end{cases}$$

The function γ^{***} is continuous in domain D^{21} since the functions γ_*^{21} and γ^{**} are continuous in the corresponding domains of definition, and their values are identical on the boundary Γ_2.

Assertion. The function γ^{***} is (u, v)-stable in the domain D^{21}.

The proof follows from the existence of saddle points in auxiliary Games 2 and 3.

Corollary 1. The optimal solution of auxiliary Game 1 consists of a series of optimal solutions of Games 2 and 3. The strategies which furnish a saddle point for Game 1 have the form

$$U_0^{(1)} = \begin{cases} U_{N^*}^{(3)}, & y(t) \neq 0 \\ U_0^{(2)}, & y(t) = 0 \end{cases},\quad V_0^{(1)} = \begin{cases} V_{N^*}^{(3)}, & y(t) \neq 0 \\ V_0^{(2)}, & y(t) = 0 \end{cases} \tag{12.1}$$

Note that the strategy $V_0^{(1)}$ in (12.1) is positional, but strategies $U_0^{(1)}$ are not positional since $U_0^{(2)}$ are the countercontrols.

The optimal trajectory of Game 1 consists of two parts. The first is the optimal trajectory of Game 3 in the time interval $t_0 \leqslant t \leqslant T_f$. It can be called a trajectory of extremal guidance to point N^*. The second part is the optimal trajectory of Game 2 over the time interval $T_f \leqslant t \leqslant T$. We shall refer to this as the trajectory of proportional pursuit, since along it the relation $z(t) / x(t) = \text{const}$ holds.

Corollary 2. The introduction of constraint (7.14) on the control of pursuers in Game 2 does not reduce the possibilities of players P_i in Game 1.

Thus, if $\{t_0, \xi(t_0)\} \in DN_1^{21}$, the optimal trajectory of Game 1 will take place for some time on the surface $y(t) = 0$, during which it emerges on it (the instant $t = T_f$), and goes down from it ($t = T$), along the tangent line ($y^{\cdot} = 0$ for $T_f^- \leqslant t \leqslant T^+$).

13. Consider the function

$$\rho^{21} = \begin{cases} \rho^{11}, & \{t_0, \xi(t_0)\} \in D^{11} \\ \gamma^{***}, & \{t_0, \xi(t_0)\} \in D^{21} \end{cases}$$

It is continuous, like the function γ^{***}, over the whole space. It was shown earlier that ρ^{11} and γ^{***} are (u, v)-stable in domains D^{11} and D^{21} respectively. Therefore, the function ρ^{21} will be (u, v)-stable over the whole space, i.e. it will be the payoff of game (1.6), (1.2), (1.7).

14. A typical trajectory of an ideal game from the initial position $\{t_0, \xi(t_0)\} \in DN_1^{21}$ is shown in Fig.5. It is a union of the optimal trajectory of Game 1 for $t_0 \leqslant t \leqslant T$ and the experimental programmed motion when $T \leqslant t \leqslant \vartheta$.

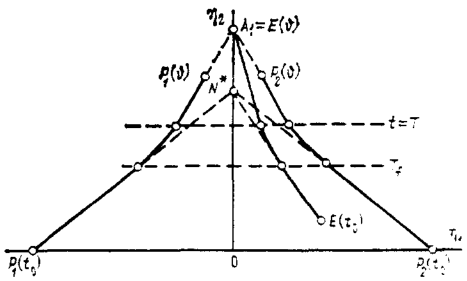

Fig.5

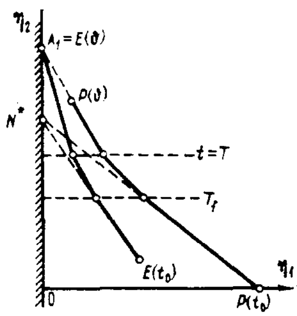

Fig.6

The set DN_2^{21} can be divided into subsets DN_{2a}^{21} and DN_{2b}^{21}. The subset DN_{2a}^{21} consists of those positions $\{t_0, \xi(t_0)\}$ for which the relations

$$\{G^i(t_0, \vartheta)\} \ni \frac{v^2}{2\mu} S_2 \supset \{G_e(t_0, \vartheta)\}$$

are standard.

We note that for such positions a one-to-one game between E and the closer pursuer, occurs that is $\rho^{21} = \rho^{11} = v^2/(2\mu)$.

We determine the set DN_{2b}^{21} as the difference of the sets $DN_{2b}^{21} = DN_2^{21} \setminus DN_{2a}^{21}$. For the initial positions $\{t_0, \xi(t_0)\} \in DN_{2b}^{21}$, the pursuers acting together ensure for themselves the result $\rho^{21} = v^2/(2\mu)$ which is better than in the one-to-one game between E and one of the pursuers. Let us look into one of these positions. We assume that players P_i and E apply the strategies $U_0^{(1)}$ and $V_0^{(1)}$. Then the corresponding trajectory will be such that at a certain instant $t = t_* \in [T_f, T]$ the points P_i and E will coincide on the η_2 axis (i.e. $z(t_*) = x(t_*) = 0$). Obviously, in such a motion Eqs.(1.6) hold to the instant $t = t_*$, since when $t > T_f$, we have $z(t)/x(t) = \text{const}$.

Note. 1°. On the basis of the payoff function constructed, given the known algorithms /5/, it is possible to formulate the physically realizable strategies which furnish the players with a result as close as desired to the payoff of a game.

2°. By virtue of the symmetries of the optimal controls obtained for players P_1 and P_2 the one-to-one problem with the phase constraints of the 'semiplane' type has an analogous solution (Fig.6).

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Translated by W.C.

PMM U.S.S.R., Vol.49, No.4, pp. 422-428, 1985 0021-8928/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

THE TRANSFORMATION OF LINEAR NON-STATIONARY OBSERVABLE AND CONTROLLABLE SYSTEMS INTO STATIONARY SYSTEMS*

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The methodological problems of the reducibility of some classes of linear non-stationary observable and controllable systems to stationary systems is considered. The constructive use of this property to analyse the controllability and observability of non-stationary systems, and also to solve applied control and estimation problems, is proposed.

For practical applications the separation of the classes of non-stationary systems, which can be investigated using simple and effective methods similar to those for analysing stationary systems, is of interest. Linear non-stationary systems for which the fundamental matrix of the solutions can be algorithmically simply constructed using the matrix of the coefficients, pertain to these calsses; in particular systems which can be reduced to stationary systems /1—5/ using the well-known non-degenerate transformation, and also systems which are Lyapunov-reducible /6, 7/. Although for non-stationary systems the sufficient conditions for controllability and observability which do not require a knowledge of the fundamental matrix of the initial system /8—10/ are known, the search for constructive transformations which reduce the initial system to a form suitable for analysing and synthesizing simple control and estimation algorithms is important and useful.

1. Consider the linear non-stationary system

$$x^{\cdot} = A(t)x + B(t)u, \quad \sigma = C(t)x \tag{1.1}$$

where x is an n-dimensional state vector of the system, u is an r-dimensional vector of the controlling action, σ is a k-dimensional vector of measurements and $A(t)$, $B(t)$, $C(t)$ are matrices of corresponding dimensions, the elements of which are continuously differentiable

*Prikl. Matem. Mekhan., 49, 4, 548-555, 1985